Aus diesen Beziehungen folgt:

Für $\varepsilon > 0$ bzw. $\varepsilon < 0$ neigt die Kette mehr zur Bildung ferro- bzw. antiferromagnetischer Bezirke. Die Ausbildung dieser Bezirke wird durch tiefe Temperaturen begünstigt. Die mittlere Größe G bzw. $\tilde{G}$ der ferro- bzw. antiferromagnetischen Bereiche ist nach (59), (61)

$$G = e^{\varepsilon/kT}(2 + e^{\varepsilon/kT}),$$
$$\tilde{G} = e^{-\varepsilon/kT}(2 + e^{-\varepsilon/kT}).$$

Für $(\varepsilon/k T) \gg 1$ wächst also die mittlere Länge der Weisschen Bezirke wie $\exp(2\varepsilon/kT)$ an. Aus (29) folgt, daß das gesamte magnetische Moment der Kette für $N \to \infty$ durch eine Gauß-Verteilung mit verschwindendem Mittelwert und der Streuung $\sigma^2 = \exp(\varepsilon/k T)$ beschrieben werden kann.

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Functional Quantum Theory of Free Relativistic Scalar Fields

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Dynamics of quantum field theory can be formulated by functional equations. Starting with the Schwinger functionals of the free scalar field, functional equations and corresponding many particle functionals are derived. To establish a complete functional quantum theory, a scalar product in functional space has to be defined as an isometric mapping of physical Hilbert space into the functional space.

The operator equations of quantum field theory can be replaced by functional equations of the corresponding Schwinger functionals [1–3]. Using this representation of quantum dynamics it is conclusive to develop a functional quantum theory with appropriate functional spaces where the complete physical information can be obtained by operations in functional spaces only [4, 5]. The most simple physical systems are the free fields. In this case the functional equations can be solved exactly as in ordinary quantum field theory and the corresponding functional space can be constructed explicitly. For the free Dirac field this has been done in [6]. In this paper the same problem is discussed for the free Klein-Gordon-field. The functional equation is solved and the space of the Schwinger functionals is equipped with a scalar product in order to reproduce the orthonormality relations of ordinary quantum field theory.

1. Fundamentals

The free scalar field is described by the Klein-Gordon-equation:

$$(\Box + m^2)\,\varphi(x) = 0\,. \tag{1.1}$$

As discussed in Appendix I, without loss of generality a neutral field can be assumed, i.e. $\varphi(x)$ is a Hermitean operator. For the quantized field $\varphi(x)$ and its derivatives $\partial^\mu \varphi(x)$ we have the following commutation relations where n denotes a timelike four vector with $n^2 = 1$ and $n^0 > 0$

$$\delta\big(n_\lambda(x^\lambda - y^\lambda)\big)[\varphi(x), \varphi(y)]_- = 0\,, \tag{1.2}$$
$$\delta\big(n_\lambda(x^\lambda - y^\lambda)\big)n_\mu[\partial^\mu \varphi(x), \varphi(y)]_- = \delta(x-y)\,.$$

Equation (1.2) are the relativistic invariant generalisations of the well known equal time commutators to the case of spacelike hypersurfaces. A derivation of (1.2) is given in Appendix I.

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As $\varphi(x)$ is supposed to describe a scalar field its transformation properties under Poincaré transformations $x' = \Lambda x + a$ are given by:

$$U \varphi(x) U^{-1} = \varphi(x') \tag{1.3}$$

where U is a representation of (Λ, a) in physical Hilbert space.

For the functional description we introduce source functions $j(x)$. As $\varphi(x)$ is a Hermitean operator the functions $j(x)$ can be chosen to be real. Furthermore their transformation properties are assumed to be [6]

$$V j(x) V^{-1} = j(x') \tag{1.4}$$

where V is a representation of (Λ, a) in the corresponding functional space. For a detailed discussion see [7].

The generating Schwinger functional is then defined by

$$\mathfrak{T}(j; a) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \langle 0 | T \varphi(x_1) \dots \varphi(x_n) | a \rangle \cdot j(x_1) \dots j(x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n . \tag{1.5}$$

T is the Wick time ordering operator which is defined invariantly according to [8] by:

$$T \varphi(x_1) \dots \varphi(x_n) = p \sum_{\lambda_1 \dots \lambda_n} \varphi(x_{\lambda_1}) \dots \varphi(x_{\lambda_n}) \, \Theta(n_\mu (x^\mu_{\lambda_1} - x^\mu_{\lambda_2})) \dots \Theta(n_\mu (x^\mu_{\lambda_{n-1}} - x^\mu_{\lambda_n})) . \tag{1.6}$$

In Eq. (1.6) n denotes the same four vector as in Eq. (1.2). As we assume the locality of the fields $\varphi(x)$, i. e. their commutativity for spacelike separations, the definition of time ordering by Eq. (1.6) is independent of the special choice of n. Following the procedure outlined in [9] the field equation (1.1) is now replaced by the functional equation for $\mathfrak{T}(j; a)$:

$$(\Box + m^2) \frac{\delta}{\delta j(x)} \mathfrak{T}(j; a) = i j(x) \mathfrak{T}(j; a) . \tag{1.7}$$

Because of the definition (1.6) of time ordering we use the commutator in the form (1.2) for the calculation of (1.7) and not at equal times as in [9]. If the invariant definition of time ordering (1.6) is applied to the free Dirac field treated in [6] we obtain the functional equation

$$(i\,{}^\mu G^\beta_\alpha \partial_\mu + m \, \delta^\beta_\alpha) \, \partial_\beta(x) \, \mathfrak{T}(j; a) = {}^5 G^\beta_\alpha j_\beta(x) \, \mathfrak{T}(j; a)$$

which clearly shows the relativistic covariance in the contrary to that derived in [6]. From the transformations properties of fields (1.3) and sources (1.4) it follows that $\mathfrak{T}(j; a)$ transforms like an invariant operator:

$$V \mathfrak{T}(j; a) V^{-1} = \mathfrak{T}'(j; a) . \tag{1.8}$$

2. Calculation of the Schwinger Functionals

In order to calculate the Schwinger functionals $\mathfrak{T}(j; a)$ it is necessary to specify the state $|a\rangle$ by its set of quantum numbers. We assume $|a\rangle$ to be a n particle state with the quantum numbers $q_1, \dots, q_n$. Especially q can mean the four momentum of the particle. From the definition of quantum numbers subsidiary conditions for the Schwinger functionals result. They are derived in [10] for the case of Fermi fields but one easily sees that they are valid for Bose fields as well. For simplicity we consider only the conservation of four momentum:

$$-i \int \mathrm{d}x \, j(x) \, \partial_\mu \frac{\delta}{\delta j(x)} \mathfrak{T}(j; a) = p_\mu \mathfrak{T}(j; a) \tag{2.1}$$

and the conservation of particle number:

$$\int \mathrm{d}x \, j(x) \frac{\delta}{\delta j(x)} \mathfrak{T}(j; a) = n \, \mathfrak{T}(j; a) . \tag{2.2}$$

In order to calculate the Schwinger functionals we first perform the transformation:

$$\mathfrak{T}(j; a) = e^{-\frac{1}{2} j \cdot F \cdot j} \quad (j; a) \tag{2.3}$$

where $j F j := \int j(x) F(x-y) j(y) \, \mathrm{d}x \, \mathrm{d}y$ and $F(x-y)$ is the two point function $\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$. (2.3) is the functional formulation of the Wick rule as is proven in Appendix II.

From (1.7) we now get the two equations:

$$(\Box + m^2) \frac{\delta}{\delta j(x)} \Phi(j; a) = 0 , \tag{2.4}$$

$$(\Box + m^2) F(x-y) = -i \, \delta(x-y) . \tag{2.5}$$

By means of (1.2) it is easily seen that

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$$

is indeed a solution of (2.5). The subsidiary conditions (2.1), (2.2) are not changed under the transformation (2.3).

For $\Phi(j; a)$ we now have the Volterra series:

$$\Phi(j; a) = \sum_{k=0}^{\infty} \frac{i^k}{k!} \int \varphi_k(x_1 \dots x_k) \, j(x_1) \dots j(x_k) \, \mathrm{d}x_1 \dots \mathrm{d}x_k . \tag{2.6}$$

[7] A. Rieckers, Z. Naturforsch. **26 a**, 631 [1971].
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By substituting (2.6) in (2.2) we get:

$$\Phi(j;a) = \frac{i^n}{n!}\cdot \int \varphi_n(x_1\dots x_n)\, j(x_1)\dots j(x_n)\,\mathrm{d}x_1\dots \mathrm{d}x_n\,. \quad (2.7)$$

The shortest way of calculating $\varphi_n(x_1,\dots,x_n)$ is its representation by the matrix element of a normal ordered product of operators:

$$\varphi(x_1,\dots,x_n) = \langle 0 | N\,\varphi(x_1)\dots\varphi(x_n) | a\rangle\,. \quad (2.8)$$

For $\varphi(x)$ we have the decomposition into creation and destruction operators:

$$\varphi(x) = \int \mathrm{d}^4 p\, \delta(p^2-m^2)\,[\langle p|\varphi(x)|0\rangle\, a^+(p) + \langle 0|\varphi(x)|p\rangle\, a(p)]\,. \quad (2.9)$$

The state $|a\rangle$ can be constructed by repeated application of a^+ on the vacuum:

$$|a\rangle = \frac{1}{\sqrt{n!}}\, a^+(p_1)\dots a^+(p_n)|0\rangle\,. \quad (2.10)$$

From the commutation relations of a and a^+, the definition of normal ordering, and $\langle 0|a^+ = 0$ we get:

$$\varphi_n(x_1,\dots,x_n) = \frac{1}{\sqrt{n!}}\, p\sum_{\lambda_1\dots\lambda_n} \langle 0|\varphi(x_1)\,|p_{\lambda_1}\rangle\dots \langle 0|\varphi(x)\,|p_{\lambda_n}\rangle\,. \quad (2.11)$$

The matrix elements $\langle 0|\varphi(x)\,|p\rangle$ can be identified with the solutions $f(x|p)$ of the one particle Klein-Gordon-equation. So the Φ-functional is finally:

$$\Phi(j;a) = \frac{i^n}{(n!)^{3/2}}\, p\sum_{\lambda_1\dots\lambda_n}\cdot \int \mathrm{d}x_1\dots\mathrm{d}x_n\, f(x_1|p_{\lambda_1})\dots f(x_n|p_{\lambda_n})\, j(x_1)\dots j(x_n)\,. \quad (2.12)$$

By the transformation (2.3) we then get immediately the Schwinger functional $\mathfrak{T}(j;a)$.

3. Functional Orthonormalization

In order to develop a complete functional quantum theory it is necesary to define a suitable scalar product in the space of the Schwinger functionals to get an isometric mapping of the physical Hilbert space into the functional space:

$$\int \mathfrak{T}^*(j;a')\,\mathfrak{T}(j;a)\,\mathrm{d}\mu(j) = \langle a'|a\rangle = \delta_{a'a}\,. \quad (3.1)$$

The left hand side of (3.1) is a functional integral [11], and we are looking for a suitable measure $\mu(j)$ to provide the orthonormality (3.1). The requirement of rotational invariance leads to [7]:

$$\mu(j) = e^{-\frac{1}{2} jGj} \quad (3.2)$$

with a symmetrical weighting function $G(x,y)$. The corect form of $G(x,y)$ has been determined for the harmonic oscillator in [12] and for the free Dirac field in [6].

The Φ-functional (2.12) is of the type of a weighted power series functional defined by:

$$D_n(x_1,\dots,x_n) = \frac{1}{n!}\, j(x_1)\dots j(x_n)\, e^{-\frac{1}{4} jGj}\,. \quad (3.3)$$

Their scalar product has been derived in [4] for an arbitrary weighting function [Eq. (I.27) of [4]]. Substituting the Φ-functionals (2.12) of two states $|a'\rangle$ and $|a\rangle$ characterized by the quantum numbers $p_1',\dots,p_m'$ and $p_1,\dots,p_n$, respectively, into this equation we get

$$\begin{aligned}\int \Phi^*(j;p_1',\dots,p_m')\,\Phi(j;p_1,\dots,p_n)\,e^{-\frac{1}{2}jGj}\,\mathrm{d}j &= p'\sum_{\lambda_1'\dots\lambda_m'} p\sum_{\lambda_1\dots\lambda_n}\sum_{\varrho=1}^{\min(n,m)} 2^{\varrho-\frac{1}{2}(n+m)}\,\frac{1}{\left(\frac{m-\varrho}{2}\right)!}\,\frac{1}{\left(\frac{n-\varrho}{2}\right)!}\,\frac{1}{\varrho!}\,\frac{1}{\sqrt{n!\,m!}}\\ &\int \mathrm{d}x_1'\dots\mathrm{d}x_m'\mathrm{d}x_1\dots\mathrm{d}x_n\, f^*(x_1'|p_{\lambda_1}')\dots f^*(x_m'|p_{\lambda_m}')\, f(x_1|p_1)\dots f(x_n|p_n)\\ &G^{-1}(x_1',x_1)\dots G^{-1}(x_\varrho',x_\varrho)\,G^{-1}(x_{\varrho+1}',x_{\varrho+2}')\dots G^{-1}(x_{m-1}',x_m')\\ &G^{-1}(x_{\varrho+1},x_{\varrho+2})\dots G^{-1}(x_{n-1},x_n)\,.\end{aligned} \quad (3.4)$$

A careful inspection of (3.4) shows that there are two types of integrals:

$$I_1 = \int f^*(x'|p')\,G^{-1}(x',x)\,f(x|p)\,\mathrm{d}x\,\mathrm{d}x', \quad (3.5\text{ a})$$

$$I_2 = \int f^*(x'|p')\,G^{-1}(x',x)\,f(x|p)\,\mathrm{d}x\,\mathrm{d}x', \quad (3.5\text{ b})$$

and the complex conjugate of (3.5 b). We now choose the weighting function such that

$$G^{-1}(x,x') = i\,n_\mu(\partial^\mu-\partial'^\mu)\,\delta(x-x')\,g(n_\lambda x^\lambda)\,g(n_\lambda x'^\lambda) \quad (3.6)$$

with a timelike four vector n as in (1.2) and a real valued function $g(n_\lambda x^\lambda)$ satisfying:

$$\int g^2(t)\,\mathrm{d}t = 1\,. \quad (3.7)$$

[11] K. O. Friedrichs and H. N. Shapiro, Seminar on Integration of Functionals, New York University, Inst. of Math. Sciences 1957.

[12] R. Weber, Z. Naturforsch. **26 a**, 220 [1971].

Then the application of (I.10), (I.11) yields for the integrals (3.5):

$$I_1 = \delta(p-p'), \quad I_2 = 0 . \tag{3.8}$$

But then we have in the sum over ϱ a nonvanishing term only for $n=m=\varrho$ because in every other term there is a factor I_2. Hence we have:

$$\int \Phi^*(j; p_1', \ldots, p_m')\, \Phi(j; p_1, \ldots, p_n e^{-\frac{1}{2} jGj}\, dj \tag{3.9}$$
$$= \frac{\delta_{nm}}{n!} p \sum_{\lambda_1 \ldots \lambda_n} \delta(p_1' - p_{\lambda_1}) \ldots \delta(p_n' - p_{\lambda_n}) .$$

(3.9) is the orthonormality relation required for the Φ-functionals. By adding the twopoint function $F(x, x')$ to the weighting function $G(x, x')$ we obtain the same orthonormality relation for the Schwinger functionals $\mathfrak{T}(j; a)$.

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Appendix I

a) Quantum Mechanical Case

In this Appendix we derive some formulas of the ordinary Klein-Gordon-theory which are needed for the evaluation of the functional scalar product in Section 3.

We first consider the one particle Klein-Gordon-equation:

$$(\Box + m^2)\, f(x) = 0 . \tag{I.1}$$

In this case $f(x)$ is a complex valued function. As the d'Alembert operator is Hermitean $f^*(x)$ is a solution of (I.1), too. There exist solutions for positive energy:

$$f^+(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta(p^2 - m^2)\, \Theta(p_0)\, \tilde{f}(p)\, e^{-ipx} \tag{I.2 a}$$

and for negative energy:

$$f^-(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p\, \delta(p^2 - m^2)\, \Theta(-p_0)\, \tilde{f}(p)\, e^{-ipx} . \tag{I.2 b}$$

By a transformation of variables it is easily seen that $f^*(x)$ is a solution for negative energy if $f(x)$ is one for positive energy.

For the solution of (I.1) a scalar product is defined by

$$(f, g) = i \int d\mathfrak{r}\, f^*(x)\, \overleftrightarrow{\partial_0}\, g(x) . \tag{I.3}$$

Specifying the solutions of (I.1) by the set of quantum numbers $p: f(x \mid p)$ we have for positive energy solutions the orthonormality relation:

$$\big(f(x \mid p'), f(x \mid p)\big) = \delta(p' - p) . \tag{I.4}$$

Because the negative energy solutions are orthogonal to the positive energy ones we have furthermore:

$$\big(f^*(x \mid p'), f(x \mid p)\big) = 0 . \tag{I.5}$$

In the following we express the scalar products (I.4), (I.5) by an integral over the whole Minkowski space as it occurs in the computation of the functional scalar product. For this reason we consider the conservation law for the current

$$j_\mu(x \mid p'\, p) = i\, f^*(x \mid p')\, \overleftrightarrow{\partial_\mu}\, f(x \mid p) \tag{I.6}$$

in its integral form:

$$\int_\Sigma j^\mu\, d\sigma_\mu = \int_{\Sigma'} j^\mu\, d\sigma_\mu \tag{I.7}$$

with arbitrary spacelike surfaces Σ, Σ'. Specializing to planes Σ: $n^\mu x_\mu - b = 0$ and Σ': $x_0 = 0$ we get from (I.4):

$$\int dx\, \delta(n^\mu x_\mu - b)\; n_\mu\, j^\mu(x \mid p'\, p) = \delta(p' - p) . \tag{I.8}$$

As b can be varied arbitrarily we may have with a function $g(b)$ fulfilling $\int g^2(b)\, db = 1$:

$$\delta(p' - p) = \int dx\, db\, g^2(b)\, \delta(n^\lambda x_\lambda - b)\; n_\mu\, j^\mu(x \mid p'\, p)$$
$$= \int dx\, g^2(n^\lambda x_\lambda)\, n_\mu\, j^\mu(x \mid p'\, p) . \tag{I.9}$$

By multiplying with $1 = \int \delta(x - x')\, dx'$ we can partially substitute x by x':

$$\delta(p' - p) = i \int dx\, dx'\, \delta(x - x') g(n_\lambda x^\lambda)\, g(n_\lambda x'^\lambda)\, n_\mu$$
$$\cdot \big(f^*(x \mid p')\, \partial'^\mu f(x' \mid p) - f(x' \mid p)\, \partial^\mu f^*(x \mid p')\big)$$
$$= i \int dx\, dx'\, f^*(x \mid p') f(x' \mid p)\, n_\mu \tag{I.10}$$
$$\cdot (\partial^\mu - \partial'^\mu)\, \big(\delta(x - x')\, g(n_\lambda x^\lambda)\, g(n_\lambda x'^\lambda)\big) .$$

(I.10) is the desired form of the orthonormality relation. By a similar treatment we get from (I.5):

$$0 = i \int dx\, dx'\, f(x \mid p')\, f(x' \mid p)\, n_\mu \tag{I.11}$$
$$\cdot (\partial^\mu - \partial'^\mu)\, \big(\delta(x - x')\, g(n_\lambda x^\lambda)\, g(n_\lambda x'^\lambda)\big) .$$

b) Field Quantized Case

In this case the function $f(x)$ is replaced by an operator $\varphi(x)$ in Hilbert space. For a neutral field $\varphi(x)$ is Hermitean and obeys the commutation relations:

$$[\varphi(x), \varphi(y)]_- = i\, \Delta(x - y) . \tag{I.12}$$

The function $\Delta(z)$ is uniquely determined by the conditions:

1) $\Delta(z)$ is an invariant solution of the Klein-Gordon-equation.

2) $\Delta(z)$ satisfies the initial conditions:

$$\Delta(0, \mathfrak{z}) = 0\,, \tag{I.13 a}$$

$$\frac{\partial}{\partial z_0} \Delta(z)|_{z_0=0} = -\delta(\mathfrak{z})\,. \tag{I.13 b}$$

From (I.13) we get the commutator on spacelike surfaces which is needed for the derivation of the functional equation in Section 1. Because of (I.13 a) we have

$$\delta\left(n_\lambda(x^\lambda - y^\lambda)\right)[\varphi(x), \varphi(y)]_- = 0 \tag{I.14}$$

as for spacelike separations $z = x - y$ there always exists a Lorentz transformation $z' = \Lambda z$ with $z_0' = 0$. By means of transformation $n' = \Lambda n$ with $n' = (1, 0, 0, 0)$ we have from (I.13 b):

$$\begin{aligned} &\delta\left(n_\lambda(x^\lambda - y^\lambda)\right) n_\mu [\partial^\mu \varphi(x), \varphi(y)]_- \\ &= \delta\left(n_\lambda'(x'^\lambda - y'^\lambda)\right) n_\mu' U^{-1} [\partial'^\mu \varphi(x'), \varphi(y')]_-\, U \\ &= i\,\delta(x'^0 - y'^0)\, U^{-1}\, \partial'^0 \Delta(x' - y')\, U \\ &= -i\,\delta(x' - y') \\ &= -i\,\delta(x - y) \end{aligned} \tag{I.15}$$

because of the invariance of the δ-function.

In the case of a charged field $\varphi(x)$ is no longer Hermitean. Then $\varphi^+(x)$ satisfies the Klein-Gordon-equation, too. But by a unitary transformation

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + i\,\varphi_2(x)\right), \tag{I.16}$$

$$\varphi^+(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) - i\,\varphi_2(x)\right).$$

Hermitean operators $\varphi_i(x)$ can be introduced. As they commute with one another the additional index can be suppressed. Thus without lack of generality we may consider a neutral field only.

Appendix II

In this Appendix we give an exact proof of Wick's theorem for the free Bose field. A similar proof is valid for free Fermi fields which is omitted here for the sake of brevity.

A free field operator $\varphi(x)$ can be decomposed into a creation operator $\varphi^-(x)$ and a destruction operator $\varphi^+(x)$:

$$\varphi(x) = \varphi^+(x) + \varphi^-(x)\,. \tag{II.1}$$

The normal ordered product of n field operators is defined by:

$$\begin{aligned} N[\varphi(x_1)\dots\varphi(x_n)] &= \varphi^-(x_1)\dots\varphi^-(x_n) \\ &+ \sum_{k=1}^{n} \varphi^-(x_1)\dots\varphi^-(x_{k-1})\varphi^-(x_{k+1})\dots\varphi^-(x_n)\,\varphi^+(x_k) \\ &+ \dots + \varphi^+(x_1)\dots\varphi^+(x_n)\,. \end{aligned} \tag{II.2}$$

From this definition we get the two basic properties of normal ordered products:

$$N[\varphi_1\dots\varphi_n\varphi^+] = N[\varphi_1\dots\varphi_n]\,\varphi^+\,, \tag{II.3 a}$$

$$N[\varphi^-\varphi_1\dots\varphi_n] = \varphi^- N[\varphi_1\dots\varphi_n]\,. \tag{II.3 b}$$

From this we get:

Lemma 1:

$$[N[\varphi_1\dots\varphi_n], \varphi_{n+1}]_- = \sum_{i=1}^{n} [\varphi_i, \varphi_{n+1}]_- N[\varphi_1\dots\varphi_{i-1}\varphi_{i+1}\dots\varphi_n]\,. \tag{II.4}$$

The proof is by induction and makes use of (II.2). It will be omitted here.

Lemma 2:

$$N[\varphi_1\dots\varphi_{n+1}] = N[\varphi_1\dots\varphi_n]\,\varphi_{n+1} - \sum_{i=1}^{n} [\varphi_i, \varphi_{n+1}^-]_-\, N[\varphi_1\dots\varphi_{i-1}\varphi_{i+1}\dots\varphi_n]\,. \tag{II.5}$$

Proof:

$$\begin{aligned} N[\varphi_1\dots\varphi_{n+1}] &= N[\varphi_1\dots\varphi_n]\,\varphi_{n+1}^+ \\ &+ \varphi_{n+1}^- N[\varphi_1\dots\varphi_n] = N[\varphi_1\dots\varphi_n]\,\varphi_{n+1} \\ &- [N[\varphi_1\dots\varphi_n], \varphi_{n+1}^-]\,. \end{aligned}$$

By application of (II.4) we get the statement.

If $t_{n+1} < t_i$, $i = 1, \dots, n$, the commutator in (II.5) is given by:

$$[\varphi_i, \varphi_{n+1}^-]_- = \langle 0 | T\,\varphi_i\,\varphi_{n+1} | 0\rangle =: F_{i,n+1}\,. \tag{II.6}$$

Now we can proof Wick's theorem connecting normal ordered and time ordered products of operators. It is:

$$T[\varphi_1\dots\varphi_n] = p \sum_{\lambda_1\dots\lambda_n} \sum_{\mu=0}^{[n/2]} \frac{1}{2^\mu\,\mu!\,(n-2\mu)!} \cdot F_{\lambda_1\lambda_2}\dots F_{\lambda_{2\mu-1}\lambda_{2\mu}} N[\varphi_{\lambda_{2\mu+1}}\dots\varphi_{\lambda_n}]\,. \tag{II.7}$$

The proof is by induction:

Without loss of generality we can assume $t_{n+1} < t_i$; $i = 1, \dots, n$, because otherwise we can change the variables until this is fulfilled. Then we have from the definition of timeordering, induction assumption, and (II.5), (II.6):

$$
\begin{aligned}
T[\varphi_1 \dots \varphi_{n+1}] &= T[\varphi_1 \dots \varphi_n]\, \varphi_{n+1} \\
&= p \sum_{\lambda_1 \dots \lambda_n}^{\lambda_1 \dots \lambda_n} \sum_{\mu=0}^{[n/2]} \frac{1}{2^\mu\, \mu!\, (n-2\,\mu)!}\, F_{\lambda_1 \lambda_2} \dots F_{\lambda_{2\mu-1}\lambda_{2\mu}} \\
&\quad \cdot N[\varphi_{\lambda_{2\mu+1}} \dots \varphi_{\lambda_n}]\, \varphi_{n+1} \\
&= p \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu=0}^{[n/2]} \frac{1}{2^\mu\, \mu!\, (n-2\,\mu)!}\, F_{\lambda_1 \lambda_2} \dots F_{\lambda_{2\mu-1}\lambda_{2\mu}}
\end{aligned}
$$

$$
\Big\{ N[\varphi_{\lambda_{2\mu+1}} \dots \varphi_{\lambda_n}\, \varphi_{n+1}]
+ \sum_{k=2\mu+1}^{n} F_{\lambda_k n+1}\, N[\varphi_{\lambda_{2\mu+1}} \dots \varphi_{\lambda_{k-1}}\, \varphi_{\lambda_{k+1}} \dots \varphi_{\lambda_n}] \Big\} .
$$

The rest of the proof runs in complete analogy to that of Lemma (II.40) in [6], and so it will be omitted here.

Expressed in terms of the generating functionals $\mathfrak{T}(j)$ and $\Phi(j)$ for the matrix elements of time ordered and normal ordered products of field operators, Wick's theorem can be formulated as:

$$\mathfrak{T}(j) = e^{-\frac{1}{2} \mathrm{jFj}}\, \varphi(j) \,. \tag{II.8}$$

The equivalence of (II.7) and (II.8) can be seen by substituting the power series expansion of $\mathfrak{T}(j)$, $\Phi(j)$, and $e^{-\frac{1}{2} jFj}$ into (II.8), and comparison of equal powers of $j(x)$. So the statement (2.3) is proven.

BERICHTIGUNGEN

Erratum to D. VOSLAMBER, Unified Model for Stark Broadening, Z. Naturforsch. **24 a**, 1458 [1969].

Page 1463: in Eq. (1) read $D^*_{\alpha'\beta'}$ instead of $D_{\alpha'\beta'}$.

Page 1466: in Eq. (17) read $R^{-1}(r - v\,\tau, v, t-\tau)$ instead of $R^{-1}(r - v\,t, v, t-\tau)$.

Page 1468: second column: in the second equation read $\overline{V}(r)$ instead of $V(r)$, and in the third equation read $-\hbar\,\omega_0$ instead of $-\omega_0$. In the last two rows of the matrix in (31) substitute b for c, and c for b.

Page 1469: the four eigenvalues mentioned in the line above the matrix (35) should be $-i\,\lambda = -i\,\sqrt{a^2+b^2+c^2}$, $+i\,\lambda$, 0, 0. Let M_{ik} denote the matrix elements of (35) ; M_{13} , M_{14} , M_{31} , and M_{41} are incorrect; the should read

$M_{13} = -(e^{i\varphi}/\sqrt{2})\,(b\,p + a\,c\,q)$, $\qquad M_{14} = (e^{-i\varphi}/\sqrt{2})\,(b\,p + a\,c\,q)$,

$M_{31} = (e^{-i\varphi}/\sqrt{2})\,(b\,p - a\,c\,q)$, $\qquad M_{41} = -(e^{i\varphi}/\sqrt{2})\,(b\,p - a\,c\,q)$.

In the matrix in (36) replace M_{21} by M_{12} , M_{22} by M_{11} , M_{11} by M_{22} , and M_{12} by M_{21} .

Page 1470: The four numerical values computed by the Monte Carlo method are too small. A correct numerical evaluation of the integral (45) and of similar expressions will be presented in a forthcoming publication.

Page 1470: Appendix A: In the first line after the second equation read $\partial\Phi_0^{(0)}/\partial t + (i/\hbar)\,\overline{H}^0\,\Phi_0^{(0)} = 0$ instead of $\partial\Phi_s^{(0)}/\partial t = 0$.

Erratum to C. BATALLI-COSMOVICI and K.-W. MICHEL, Barium Exhaust Spectrum during the Expansion of Combustion Products through a Nozzle into Vacuum, Z. Naturforsch. **26 a,** 1147 [1971].

On page 1152 a, Fig. 7, we have to add to the figure caption:

a) 10 msec after ignition,
b) 50 msec after ignition.

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